Received: *December 17, 1999*, Accepted: *January 4, 2000*

Multivalued entropy of supersymmetric black holes

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Abstract: The supersymmetric flow equations describing the flow of moduli from infinity to the black hole horizon, and vice versa, are derived in the five-dimensional theories where the moduli space of the very special geometry has disjoint branches. The multiple solutions are derived from the 'off the horizon' attractor equation. Within each branch, the black hole entropy, as usual, depends only on the near horizon attractor values of moduli, i.e. the entropy depends on the charges and on coefficients of the cubic polynomial. It does not depend on the values of the moduli fields at infinity. However, the entropy, as well as the near horizon values of the moduli fields, are shown to depend on the choice of the branch specified by the choice of the set of moduli at infinity. We present examples of BPS black hole solutions with the same Q_I and C_{IJK}, whose entropies differ significantly.

Keywords: Superstring Vacua, Black Holes in String Theory.

[*]On leave of absence from Stanford University until 1 September 2000.

During the last decade supersymmetric black holes played an important role in setting up the issues of the fundamental theory, including gravity. For example, an important property of the supersymmetric black holes is the manifest symmetry under U-duality transformations. The explanation of this fact from the point of view of type-IIA string theory requires first to promote it to an 11-dimensional theory and afterwards to compactify it on a torus. Not much is known about 11-dimensional M theory, but the existence of U-duality invariant BPS states, like supersymmetric black holes, helps to explore the non-perturbative string/M-theory.

The purpose of this note is to present some new, previously unexplored features of supersymmetric black hole entropy: its non-uniqueness in theories with disjoint branches of the moduli space. The existence of multiple critical points relevant to BPS black holes in the theories with the same electric charges but disjoint branches of moduli space was already established before [1].

The issue of the non-uniqueness of the supersymmetric black hole entropy was raised in [2] in the context of black holes of Calabi-Yau spaces. Here we consider arbitrary $d = 5$, $N = 2$ supergravity theory [3], the moduli space is not symmetric, in general. We put no restrictions on Chern-Simons couplings C_{IJK}: the supersymmetry is valid for all these generic theories. In some cases they may be interpreted as Calabi-Yau intersection numbers. The uniqueness of the entropy in the theories where the moduli space has only one branch with the positive metric to large extent follows from the fact, established in [4], that the entropy is a minimum of the BPS mass.[1] The existence of disjoint branches of the moduli space of five-dimensional supergravity [3] was first pointed out in [6].

In the context of gauged supergravity the disjoint branches of moduli space were studied in [7, 1]. An important feature of these branches is that not only the metric of scalar fields [7] but also the metric of vector fields is positive-definite [1], which means that these branches are quite legitimate. In particular, we have found different AdS_5 branches with equal values of the cosmological constant, which is one of the necessary conditions for realization of a supersymmetric generalization of the one-brane Randall-Sundrum scenario [8]. However, we were able to show (see also [7]) that all interpolating domain wall solutions (not only BPS states) in a broad class of supersymmetric models studied in [1] are not of the Randall-Sundrum type.

In this paper we will present the complete black hole solutions, i.e. define the moduli and the metric and the vector fields of the black holes everywhere, not only at the critical point near the horizon. There will be at least two solutions depending on the same harmonic functions, one with the central charge Z positive and the other with the central charge Z negative. This will explain the multivalued nature of the black hole entropy.

[1] Recently a detailed derivation of the uniqueness of the BPS black hole entropy was performed in [5]. The claim was also made that the entropy is always unique. The argument, however, is based on an assumption that the moduli space consists of a single branch.

We study the five dimensional $N = 2$ ungauged supergravity interacting with n abelian vector multiplets [3, 9]. The supersymmetric black hole solutions of this theory in the context of the very special geometry were studied in [10, 11]. A general ansatz for the black hole solution with positive central charge Z was proposed in [12]. The ansatz is given in terms of $n+1$ harmonic functions, $K_I = k_I + Q_I/r^2$. It provides an explicit black hole solution only in cases when the solution of the stabilization-type equations $C_{IJK}Y^JY^K = K_I$ is available. Stabilization equations in general are known to define the values of the moduli near the black hole horizon [13, 14]. The specific form of stabilization equations in $N = 2$ $d = 5$ supergravity interacting with vector multiplets, $C_{IJK}Y^JY^K = Q_I$, was found in [11]. It gives the values of the fixed scalars near the horizon. An analogous equation is also a part of the stabilization equations in $d = 4$ $N = 2$ theory with a cubic prepotential [15] relevant to Calabi-Yau black holes.

We will present below an ansatz for the black hole solutions with both positive and negative central charges $Z = X^I(\phi)Q_I$. The surprising feature of it is that sometimes both solutions with positive and negative central charges (graviphoton charges) occur for the same choice of harmonic functions and in particular for the same individual vector fields charges Q_I.

The reason why it is surprising to have a central charge, i.e. the graviphoton charge, both positive and negative for the same charges of the vector fields Q_I is the following. The graviphoton charge is given by the moduli dependent combination $Z = X^I(\phi)Q_I$ of the individual vector fields charges Q_I. If there are no vector multiplets and $I = 0$, the graviphoton charge is equal to the usual charge $Z = Q_0$. The central charge is positive for positive Q_0 and negative for negative Q_0. In the first case we have $M = Z$, in the second case $M = -Z$. In presence of moduli, as shown in [1] it is possible to have both values of Z without changing the sign of Q_I. After having observed this unusual situation near the critical point we would like to show the full solution with such properties.

The very special geometry of five-dimensional supergravity emerges because the independent moduli ϕ^i, $i = 1, \ldots, n$ are coordinates describing some cubic hypersurface

$$\mathcal{V} = \frac{1}{6} C_{IJK}X^IX^JX^K = 1\,, \qquad I = 0, 1, \ldots, n\,. \tag{1}$$

The five-dimensional bosonic $N = 2$ lagrangean is:

$$e^{-1}\mathcal{L} = -\frac{1}{2}R - \frac{1}{4}G_{IJ}F_{\mu\nu}{}^IF^{\mu\nu J} - \frac{1}{2}g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j + \frac{e^{-1}}{48}\epsilon^{\mu\nu\rho\sigma\lambda}C_{IJK}F^I_{\mu\nu}F^J_{\rho\sigma}A^k_\lambda\,. \tag{2}$$

The gauge coupling metric G_{IJ} and the moduli space metric g_{ij} are

$$G_{IJ}(\phi) = -\frac{1}{2}\frac{\partial}{\partial X^I}\frac{\partial}{\partial X^J}(\ln\mathcal{V})|_{\mathcal{V}=1}\,, \qquad g_{ij}(\phi) = G_{IJ}\partial_iX^I\partial_jX^J|_{\mathcal{V}=1}\,. \tag{3}$$

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We are looking for the electrically charges black holes with the metric

$$ds^2 = -e^{-4U}dt^2 + e^{2U}(d\vec{x})^2 \,. \tag{4}$$

The Chern-Simons term does not contribute to electric configuration where only F^I_{0r} are not vanishing. We introduce a function $\mathcal{Z} \equiv X^I G_{IJ} F^J_{or}$ and a function $\mathcal{Z}_i \equiv \partial_i X^I G_{IJ} F^J_{or}$ and consider configurations for which $\mathcal{Z}_i = \partial_i \mathcal{Z}$. The action can be rewritten in a nice BPS form.

$$\begin{aligned} E &= -\int dr^{+\infty}_{-\infty} \mathcal{L} \qquad (5) \\ &= \int_{-\infty}^{+\infty} dr \left[\frac{3}{2}\left(\frac{\partial U}{\partial r} \pm \frac{1}{3} e^{2U}\mathcal{Z}\right)^2 + \frac{1}{2}\left(\frac{f_{ai}\partial\phi^i}{\partial r} \pm \frac{1}{2}e^{2U} f^i_a \mathcal{Z}_i\right)^2 \mp \frac{1}{2}\frac{\partial}{\partial r}\left(e^{2U}\mathcal{Z}\right)\right]. \end{aligned}$$

Here we used the moduli space vielbein f^a_i where $f^a_i f^b_j \delta_{ab} = g_{ij}$. The analogous form of the action in case of 4-dimensional black holes was found in [4]. One can use either supersymmetry or just very special geometry to derive eq. (5).

The first order flow equations which define the evolution of the metric function $U(r)$ and of the moduli $\phi^i(r)$ follow from this action

$$\frac{\partial U}{\partial r} \pm \frac{1}{3} e^{2U}\mathcal{Z} = 0 \,, \qquad \frac{g_{ij}\partial\phi^j}{\partial r} \pm \frac{1}{2} e^{2U}\mathcal{Z}_i = 0 \,. \tag{6}$$

Solutions of these equations with the boundary conditions for which the surface term vanishes saturate the BPS bound. The solutions are given by

$$\begin{aligned} ds^2 &= -e^{-4U}dt^2 + e^{2U}(d\vec{x})^2 \,, \\ G_{IJ}F^I_{0m} &= \frac{1}{4} e^{-4U}\partial_m K_J \,, \\ e^{2U} &= \pm \frac{1}{6} X^I K_I \,. \end{aligned} \tag{7}$$

Here K_I is a harmonic function,

$$K_I = k_I + \frac{Q_I}{r^2} \,, \tag{8}$$

and the moduli $X^I(\phi)$ are real and have to satisfy the 'off the horizon' attractor equation

$$\pm e^{2U} C_{IJK} X^J X^K = K_I \,. \tag{9}$$

The ansatz with the upper sign is the one found by Sabra [12]. The appearance of the second solution with the minus sign for the same choice of the harmonic function is new since we are not changing K_I to $-K_I$. It is clear from eq. (7) that e^{2U} must be everywhere positive since it is a component of the space-time metric. Still the combination $X^I K_I$ can be either positive or negative: only in such cases our ansatz gives a consistent solution. Both cases may exist as we will see soon.

Near the black hole horizon our 'off the horizon' attractor equation (9) reduces to the near horizon attractor equation

$$\frac{Z_{hor}}{6} C_{IJK} X^J X^K = Q_I \,. \tag{10}$$

To see this one has to use the fact that at $r \to 0$ the metric tends to $e^{2U} \to |Z|_{hor}/6r^2$.

If we absorb the central charge into the redefinition of the moduli near the horizon, $\bar{X}^I = \sqrt{Z_{hor}/6}\, X^I$, we may bring the near horizon stabilization equation to the form

$$C_{IJK} \bar{X}^I_{hor} \bar{X}^J_{hor} = Q_I \,. \tag{11}$$

In this form it was studied extensively in [11] and, for the multivalued central charge Z, in [1].

The multivalued nature of the black hole entropy was discussed in [1] on the basis of the near horizon attractor equation (11). At least 2 different solutions may exist: one with positive Z and another one with negative Z. The fields $\bar{X}^I$ may be real and imaginary, in what follows we will denote them by Y^I_{re} and Y^I_{im}, respectively.

$$\sqrt{+|Z_{hor})|/6}\, X^I_{hor} = Y^I_{re}(r=0) \,, \qquad \sqrt{-|Z_{hor}|/6}\, X^I_{hor} = Y^I_{im}(r=0) \,. \tag{12}$$

In the first case the redefined field is real, in the second case it is imaginary. However, the moduli X^I in both cases is real. The near horizon equations take the form

$$C_{IJK} Y^J_{re} Y^K_{re} = Q_I \,, \qquad C_{IJK} Y^J_{im} Y^K_{im} = Q_I \,. \tag{13}$$

Even more solutions may exist in many-moduli case, particularly if the stabilization equations (11) have more solutions for which the metric of the moduli space and the one for the vector space are positive-definite. We will see now how all of this generalizes for the full black hole solution. First we redefine the fields in eq. (9). We introduce $\sqrt{\pm e^{2U}} X^I = \bar{X}^I(r)$. In terms of these variables the 'off the horizon' attractor equation takes the form:

$$C_{IJK} \bar{X}^J(r) \bar{X}^K(r) = K_I(r) \,. \tag{14}$$

Introduce

$$\sqrt{+e^{2U}} X^I = Y^I_{re}(r) \,, \qquad \sqrt{-e^{2U}} X^I = Y^I_{im}(r) \,. \tag{15}$$

The off the horizon attractor equations take the form

$$C_{IJK} Y^J_{re}(r) Y^K_{re}(r) = K_I(r) \,, \qquad C_{IJK} Y^J_{im}(r) Y^K_{im}(r) = K_I(r) \,. \tag{16}$$

Here again we will be looking for the solutions of the stabilization equations with both real and imaginary Y^I which correspond to real X^I. Note that after the field redefinition the off the horizon equations (16) look exactly as the near horizon equations (13) but now the moduli $Y(r)$ are defined by its solutions via $K_I(r)$

everywhere at all values of r and not only at $r = 0$. Still the solutions of the attractor equations for $Y^I_{re}(r)$ and $Y^I_{im}(r)$ in terms of the harmonic functions $K_I(r)$ are precisely the same as in the near horizon cases the solutions for $Y^I_{re}(r = 0)$ and $Y^I_{im}(r = 0)$ in terms of the constants Q_I. This means that in all cases when these equations near horizon were solved (or will be solved eventually), and multiple solutions are available near the horizon, the multivalued black hole solution are obtained simply by replacing the constants Q_I with the harmonic functions $K_I(r)$. In particular, using examples of solutions with both real and imaginary values of $Y^I(r = 0)$ given in [1], we will find below the full black hole solutions with both real and imaginary $Y^I(r)$ and two values of real $X^I(r)$.

Thus, to find the black hole solutions in these theories one should explicitly solve the algebraic attractor equations. The metric will be obtained straightforwardly from such solutions.

There is an interesting property of our attractor equations at $r \to \infty$. Since the parameters of the solutions are given by harmonic functions H_I the values of moduli at $r \to \infty$ are not free: rather we have to solve the stabilization equation at $r \to \infty$ to find the values of the moduli there.

$$C_{IJK}(Y^J_{re} Y^K_{re})_{r\to\infty} = k_I \,, \qquad C_{IJK}(Y^J_{im} Y^K_{im})_{r\to\infty} = k_I \,. \tag{17}$$

There is a restriction on the choice of k_I for a given C_{IJK} such that for both solutions $e^{2U} \to 1$ at $r \to \infty$:

$$e^{2U}_{r\to\infty} = \pm\frac{1}{6}(X^I_{r\to\infty})\; k_I = 1\,. \tag{18}$$

The double extreme black holes are the ones with constant moduli. To find such we may choose $k_I = \alpha Q_I$ and all harmonic functions will have the factorizable dependence on r of the form $K_I = k_I(1 + \alpha/r^2)$. The moduli fields if chosen e.g. as the ratio of X^I/X^0 will be r-independent for such solutions and the metric will be $e^{2U} = (1 + \alpha/r^2)$. We will find examples of such double-extreme black holes which live in the two different branches of the moduli space. The values of moduli will be different in two branches, but the metric and therefore the entropy will be the same.

For the black hole solutions with non-constant moduli, the value of the metric at infinity defined by the choice of h_I will be the same. However at the horizon the metric in two different branches of the moduli space and, consequently, the black hole entropy, can be significantly different.

The theory with one independent moduli is relatively easy to understand. It may give some insights into the properties of the black holes in more general and more interesting cases with many moduli, in particular, related to Calabi-Yau spaces. The positive definiteness of the moduli space metric and the gauge couplings has been analyzed in one-moduli case in [1]. In more general situations this will be a more complicated problem.

Consider a simple case of $I = 1, 2$ and generic C_{IJK} and $H_I = \frac{1}{2}K_I$. According to the discussions above, we have to take our solutions of the near horizon attractor equations in [1] and replace the charges there by the harmonic functions $K_I(r)$ to get the full metric. It gives for the two cases of Y^I_{re} and Y^I_{im} the solutions for the metric and for the moduli $\phi \equiv X^2/X^1$:

$$
\begin{aligned}
\mp \left(e^{6U}(r)\right)_{P/M} &= -\frac{K_2(dK_1^2 + bK_2^2 - 2cK_1K_2)}{36M} + \\
&\quad + \mathcal{D}\left[\frac{F(K)L + 2E(K)M \pm \sqrt{4M^2\mathcal{D}(K)}}{36M(L^2 - 4MN)}\right], \qquad (19) \\
\left(\phi(r)\right)_{P/M} &= \frac{-E(K) \pm \sqrt{\mathcal{D}(K)}}{F(K)}. \qquad (20)
\end{aligned}
$$

Here[2]

$$
\begin{gathered}
C_{111} = a\,, \quad C_{112} = b\,, \quad C_{122} = c\,, \quad C_{222} = d\,, \\
M \equiv c^2 - bd\,, \qquad N \equiv b^2 - ac\,, \qquad L \equiv ad - bc\,, \\
\mathcal{D}(K) \equiv (MK_1^2 + NK_2^2 + LK_1K_2)\,, \\
E(K) \equiv cK_1 - bK_2\,, \qquad F(K) \equiv dK_1 - cK_2\,. \qquad (21)
\end{gathered}
$$

The intermediate expressions for Y^1_{re}, Y^2_{re} and Y^1_{im}, Y^2_{im} can be found using [1] and performing the replacement of charges by harmonic functions. We derived the solutions above using the following definitions:

$$
\phi_P = \left(\frac{X^2}{X^1}\right)_P = \frac{Y^2_{im}}{Y^1_{im}}\,, \qquad \phi_M = \left(\frac{X^2}{X^1}\right)_M = \frac{Y^2_{re}}{Y^1_{re}}\,, \qquad (22)
$$

and

$$
\left(e^{6U}(r)\right)_P = -\left(\frac{1}{6}Y^I_{im}K_I\right)^2, \qquad \left(e^{6U}(r)\right)_M = \left(\frac{1}{6}Y^I_{re}K_I\right)^2. \qquad (23)
$$

The moduli space metric is

$$
g_{\phi\phi} = \frac{3[N - L\phi + M\phi^2]}{[a + 3b\phi + 3c\phi^2 + d\phi^3]^2}\,. \qquad (24)
$$

We assume that $L^2 - 4MN < 0$ and $M > 0$, $N > 0$ so that the metric of the moduli space is positive. The metric, however, may have singularities which show that the moduli space defined by the constrained surface $\mathcal{V} = 1$ has disjoint branches: the multivalued black hole solutions exist in each separate branch of the moduli space. The study of the vector space metric (gauge couplings) for the black holes defined above is performed in analogy with the near horizon case in [1] and we find again that it is positive.

[2]We assume that $M \neq 0$ and $L^2 \neq 4MN$.

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The fixed values of moduli at infinity and at the horizon are

$$(\phi(r))^{\infty}_{P/M} = \frac{-E(k) \pm \sqrt{\mathcal{D}(k)}}{F(k)}\,, \qquad (\phi(r))^{hor}_{P/M} = \frac{-E(Q) \pm \sqrt{\mathcal{D}(Q)}}{F(Q)}\,. \tag{25}$$

For arbitrary choice of harmonic functions the values of $X^I K_I$ for two solutions (with Y_{re} and Y_{im}) are different. But at $r \to \infty$ the metric has to be the same, $e^{2U}_{r\to\infty} = 1$. Fortunately, solutions of such type have been found in [1]. One has to require, therefore, that by solving the attractor equations at infinity (17) one gets the same value of $X^I K_I$ in case of Y_{re} and Y_{im}. The equation for our parameters specifying this case is:

$$\mathcal{A} \equiv -k_2(dk_1^2 + bk_2^2 - 2ck_1h_2) + \mathcal{D}(k)\left[\frac{F(k)L + 2E(k)M}{(L^2 - 4MN)}\right] = 0\,. \tag{26}$$

In [1] we have found several families of such solutions of the stabilization equations. We will use these examples to satisfy the condition $\left(e^{2U}\right)_{P/M} = 1$ at $r \to \infty$.

To show that the analytic solutions for the multivalued black holes have non-trivial examples and to understand the properties of such solutions, we plot some of them for a particular choice of C_{IJK}. As the purpose of this paper is to promote the multiple critical points of our attractor systems to complete black hole solutions, we will use the same one-moduli theory with $a = 0,\ b = 1/3,\ c = 4/3,\ d = 1$ which we used in [1] and where we checked that the physical conditions of the positivity of the moduli space metric and of the gauge coupling matrix are satisfied. In this case the moduli space metric is singular at $\phi = 0, \phi \sim -0.27, \phi \sim -3.73$. It is positive everywhere but discontinuous. Therefore the expectation is that if the moduli at infinity starts in one of the branches, it will flow to the black hole horizon remaining in the same stripe of the moduli space, so that at all r the black hole moduli $\phi(r)$ is inside of a given branch where it started and the metric g_{tt} is smooth all the way from infinity to the horizon. This indeed is a property of our solutions. We plot some examples of multivalued black hole solutions and their entropy. In all our examples we take $k_1 = 2,\ k_2 = 4$. These provide the correct asymptotic behavior of the metric $e^{2U} \to 1$ and keep the initial values of moduli in two disconnected branches of the moduli space: the first one in the branch $0 < \phi < +\infty$ and the second one in the branch $-0.27 < \phi < 0$. In all examples we will plot the moduli $\phi(r)$ and the space-time metric. We also plot the metric $g_{tt}(r) = e^{-4U(r)}$; this function tends to 0 near the horizon at $r = 0$ and tends to 1 at $r \to \infty$. To find the entropy we plot for each example the value of $r^3 e^{3U(r)} = |Z_{hor}/6|^{3/2}(r) = |\tilde{Z}_{hor}|^{3/2}(r)$ near the horizon. The black hole entropy is proportional to $|\tilde{Z}_{hor}|^{3/2}(0)$.

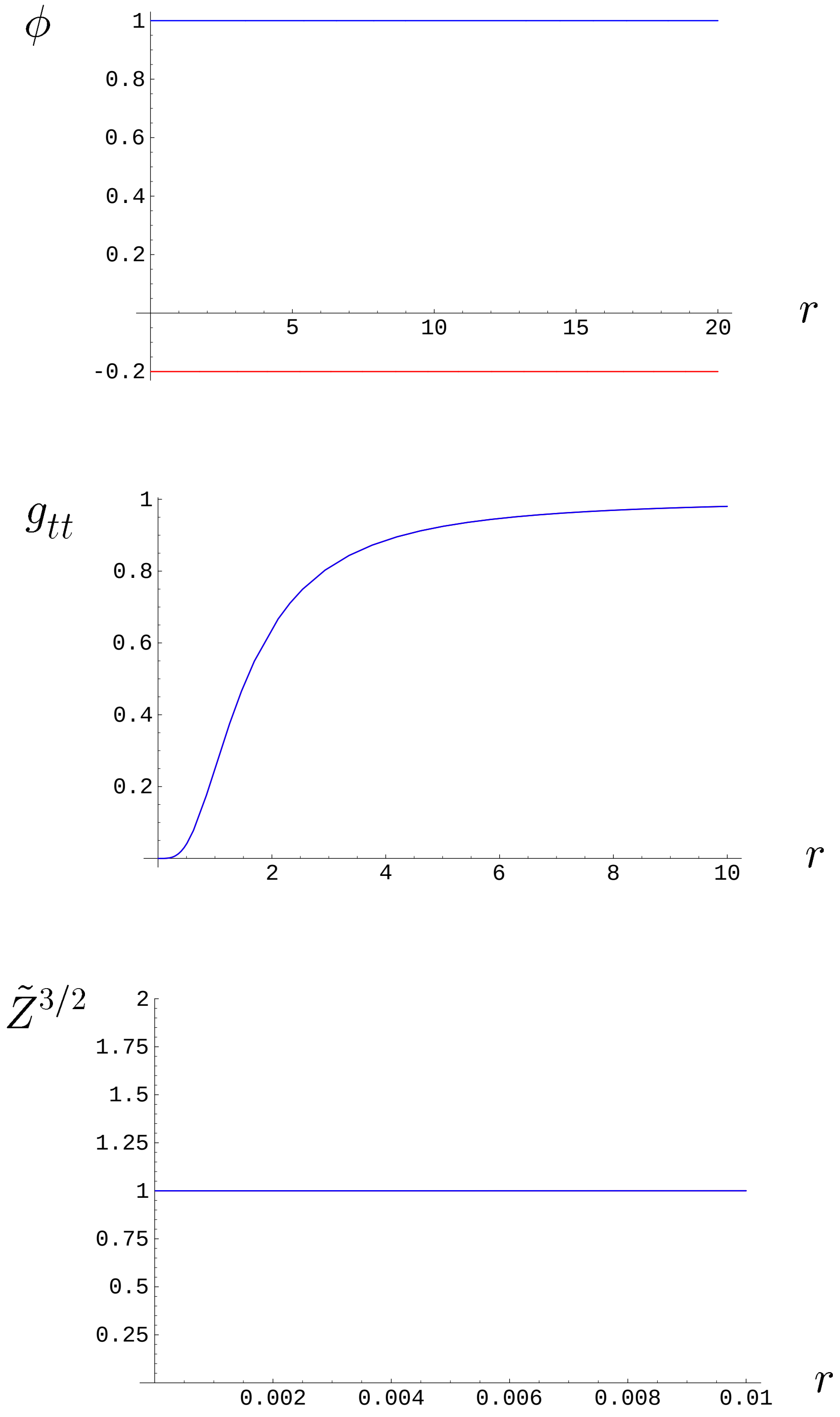

Figure 1: An example of a multivalued double extreme black hole with everywhere constant moduli, which are different in two branches of the moduli space. The metric (and therefore the entropy) for both choices of moduli are the same.

It is useful to start with the double extreme black holes: for them the moduli does not change, so it definitely stays in the same branch where it started. We make here the simplest choice of harmonic functions $K_I = k_I(1 + 1/r^2)$ with $k_1 = 2$, $k_2 = 4$. The plot of moduli ϕ in figure 1 shows that one of them remains equal to $+1$ everywhere and the other one remains equal to -0.2. In figure 1 we also show the metric: we find that g_{tt}, is the same for both solutions, as follows from analytic expression for the double extreme black holes. The entropy is also the same for these two solutions.

The second example which we plot in figure 2 is for a choice $k_1 = 2$, $k_2 = 4$, $Q_1 = 0.49$, $Q_2 = 1.8275$. This particular choice of charges was taken simply to show that one can find a solution with the entropies near the horizon of each black hole different 10^3 times. In figure 2 we plot the two flows of moduli, each in its branch of the moduli space. To show that both solutions are nice and smooth, we plot $g_{tt}(r))_{P/M} = (e^{-4U(r)})_{P/M}$ on two different scales: one shows that both metrics at infinity approach 1, the second is closer to the horizon. We also plot the entropy. We have to make two different plots, since the first one differs from the second one 10^3 times. We have plotted many other examples with simple values of charges, like 1, 3, 5 etc. In such cases we found the entropies in two branches differ moderately, like 2 or 10 times.

Thus we have studied supersymmetric black holes in d=5, N=2 supergravity in cases when the moduli space may have disjoint branches with everywhere positive metric. We have found black hole solutions in each branch which are different despite the black hole charges and the cubic surface defining a supergravity theory are the same. The new supersymmetric black hole entropy formula is

$$S = S(Q_I, C_{IJK}, N_l)\,, \tag{27}$$

where the dependence on N_l with $l = 1 \ldots, k$ indicates the dependence on the branch of the moduli space in case there are k branches. It would be interesting to find the multivalued black hole solutions for some parameters which may appear within a context of string/M-theory.

It remains to find out whether this observation of the unusual properties of supergravity black holes can be used to probe the properties of the fundamental theory.

Acknowledgments

I am grateful to A. Chamseddine, S. Ferrara, A. Linde, G. Moore, M. Shmakova, and J. Rahmfeld and especially to G. Gibbons and A. Van Proeyen for valuable discussions. I am grateful for the hospitality to the Institute of Theoretical Physics in Leuven where this work was finished. This work was supported in part by NSF grant PHY-9870115.

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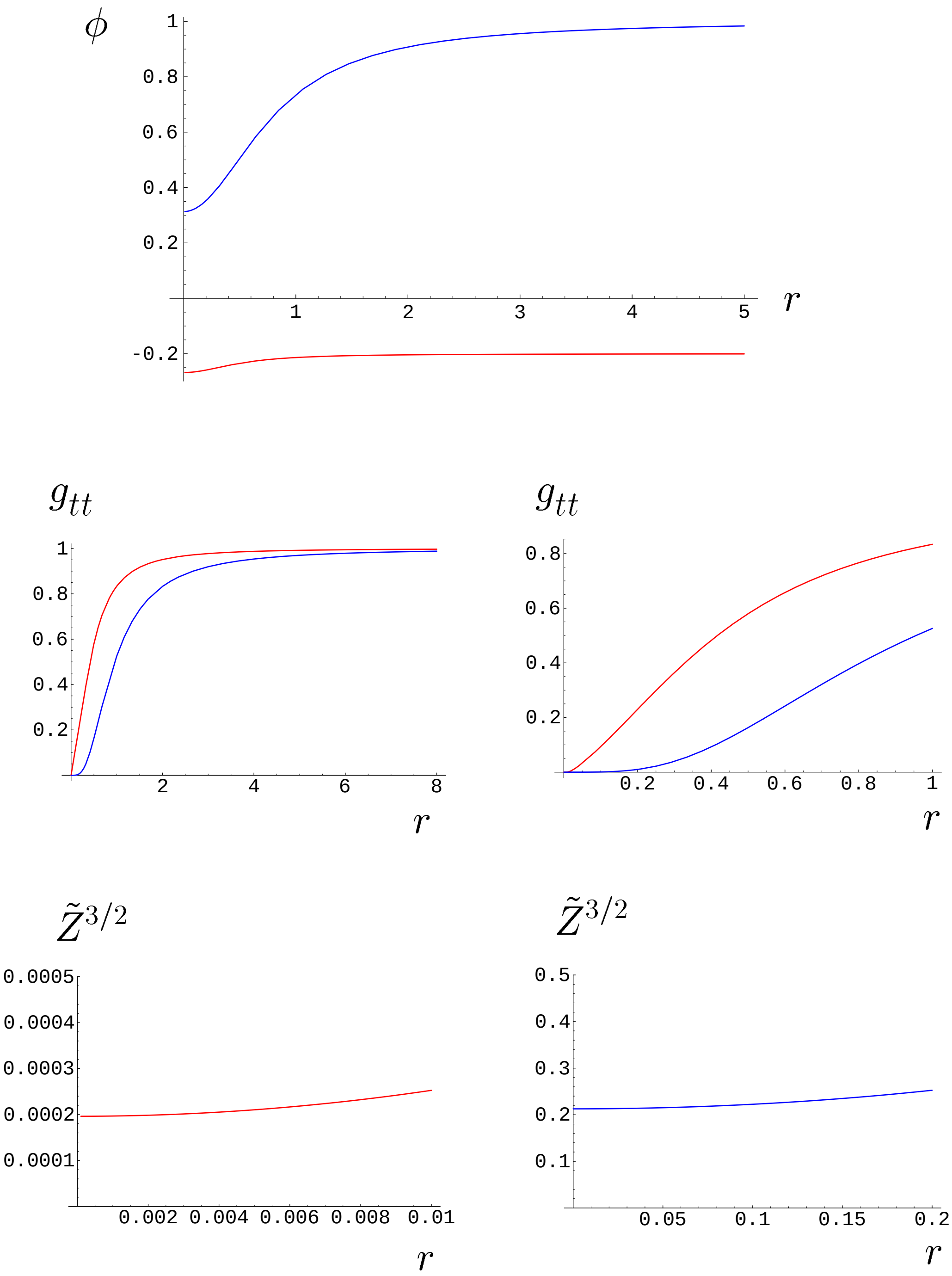

Figure 2: An example of a black hole solutions with two attractors, one for each branch of the moduli space. The metric and the entropy differ strongly near the horizon. The entropy of these black holes differs by 10^3 times.

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JHEP01(2000)001